Linear Algebra in Combinatorics – Problems

Krit Boonsiriseth

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1 More Linear Algebra facts

Before we get started with the problems, here are some facts that might come in handy.

- Given a subspace W of a vector space V with a dot product, the set
$$W^{\perp} := \{\vec{v} \in V : \vec{v} \cdot \vec{w} = 0 \text{ for all } w \in W\}$$
forms a subspace of V, and $\dim W + \dim W^{\perp} = \dim V$.
- The **determinant** of a square matrix A can be evaluated by taking a sum over all permutations σ of $\{1, 2, ..., n\}$:
$$\det A = \sum_{\sigma \in S_n} (-1)^{\operatorname{sgn} \sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdot a_{n\sigma(n)},$$
where the **sign** of a permutation is equal to the number of inversions in it (i.e. the number of $i < j$ such that $\sigma(i) > \sigma(j)$). In the field of two elements $\mathbb{F}_2$ (think of it as integers modulo 2), $-1 = +1$, so we don't have to worry about the sign.
- The following are all equivalent, for square matrices A:
  i. A is invertible.
  ii. $\det A$ is nonzero.
  iii. A has full rank.
- A matrix of the form $tI + D$, where $t \geq 0$ and D is a diagonal matrix with positive entries, always has full rank.

2 Oddtown, Eventown, and all that jazz

These problems generally involve techniques similar to the Oddtown problem we discussed in class. You should be able to do most of these (and especially Problem 5) by following the proof of Oddtown. Problems marked with a dagger (†) are more technical / require extra background on linear algebra than what was presented today.

1. There are $2n$ people at a party. Each person has an even number of friends at the party. (Here, friendship is a mutual relationship.) Prove that there are two people who have an even number of common friends at the party.
2. At another party, every pair of two people have an odd number of common friends. Show that there are odd number of people at this party.

3. A set T is called even if it has an even number of elements. Let n be a positive even integer, and let $S_1, \dots, S_n$ be even subsets of the set $\{1, \dots, n\}$. Prove that there exist some $i \neq j$ such that $S_i \cap S_j$ is even.

4. (Oddtown and Eventown) In a certain town with n citizens, a number of clubs are set up. No two clubs have exactly the same set of members. Determine the maximum number of clubs that can be formed under each of the following constraints:

   a. The size of every club is odd, and every pair of distinct clubs share an even number of members.

   b. The size of every club is even, and every pair of distinct clubs share an odd number of members.

   † c. The size of every club is even, and every pair of distinct clubs share an even number of members.

   † d. The size of every club is odd, and every pair of distinct clubs share an odd number of members.

5. Let $A_1, \dots, A_m$ and $B_1, \dots, B_m$ be subsets of $\{1, \dots, n\}$ satisfying the following two conditions.

   (i) $|A_i \cap B_i|$ is odd for all $i = 1, \dots, m$.

   (ii) $|A_i \cap B_j|$ is even for all $1 \leq i < j \leq m$.

   Prove that $m \leq n$.

6. (Fisher's inequality) Let $C_1, \dots, C_m \subseteq \{1, 2, \dots, n\}$ be distinct nonempty subsets, such that all pairwise intersections $C_i \cap C_j$ (for $i \neq j$) have the same size. Show that $m \leq n$.

7. Students in a school go for ice cream in groups of at least two. After $k > 1$ groups have gone, every two students have gone together exactly once. Prove that the number of students in the school is at most k.

3 Geometry in $\mathbb{R}^n$

The **norm** (or length) of a vector $\vec{v} = [v_1, \dots, v_n]$ in $\mathbb{R}^n$ is defined to be

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

The **distance** between two vectors $\vec{v} = [v_1, \dots, v_n]$ and $\vec{w} = [w_1, \dots, w_n]$ in $\mathbb{R}^n$ is defined to be

$$\|\vec{v} - \vec{w}\| = \sqrt{(v_1 - w_1)^2 + \cdots + (v_n - w_n)^2}.$$

The **angle** θ between two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ is defined by

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}.$$

8. Given a positive integer d, show that there cannot be more than $d + 1$ points in $\mathbb{R}^d$ such that the distances between any two of these points are equal.

† 9. (Equiangular lines) There are at most $\binom{d+1}{2}$ lines through the origin in $\mathbb{R}^d$ such that the angle between any two of them is the same.

† 10. (Two-distance sets) Let $p_1, \dots, p_n \in \mathbb{R}^d$ be distinct points such that there are only two distinct distances between them. Prove that $n \leq \binom{d+2}{2}$.

4 Adjacency matrices

The **adjacency matrix** of a graph is a matrix $A = \{a_{ij}\}_{i,j \in V(G)}$ whose rows and columns are indexed by the vertices of the graph, and with

$$a_{ij} = \begin{cases} 1 & \text{if } ij \text{ is an edge in } G \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is symmetric for simple graphs, because ij is an edge iff ji is an edge. We can likewise define adjacency matrices for *directed* graphs, but the resulting matrix will not necessarily be symmetric.

11. Let G be a graph on the vertex set $\{1, \dots, n\}$ and let A be its adjacency matrix. Prove that for any $i, j \in \{1, \dots, n\}$ and any non-negative integer m, the (i, j)-entry of the matrix A^m equals the number of walks of length m in G from vertex i to vertex j.

12. (Putnam 2022/B5) Say that an n-by-n matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ with integer entries is *very odd* if, for every nonempty subset S of $\{1, 2, \dots, n\}$, the $|S|$-by-$|S|$ submatrix $(a_{ij})_{i,j \in S}$ has odd determinant. Prove that if A is very odd, then A^k is very odd for every $k \geq 1$.

It turns out that most of the fun stuff involving adjacency matrices (and also a lot of the fun stuff about linear algebra) will involve what are called **eigenvectors** and **eigenvalues**, but this is a bit beyond the scope of any olympiad problem.

5 More Problems

These are problems that are mostly taken from actual contests, and might involve anywhere from zero to a lot of linear algebra. It will be useful to think of things in a linear algebra mindset though.

13. (Classical) Let $a_1, a_2, \dots, a_{2n+1}$ be real numbers, such that for any $1 \leq i \leq 2n + 1$, we can remove a_i and separate the remaining $2n$ numbers into two groups of n numbers with equal sums. Show that $a_1 = a_2 = \cdots = a_{2n+1}$.

14. (Russia 2001) A contest with n question was taken by m contestants. Each question was worth a certain (positive) number of points, and no partial credits were given. After all the papers have been graded, it was noticed that by reassigning the scores of the questions, any desired ranking of the contestants could be achieved. What is the largest possible value of m?

15. (Iran 1996) Let G be a finite simple graph, and there is a light bulb at each vertex of G. Initially, all the lights are off. Each step we are allowed to chose a vertex and toggle the light at that vertex as well as all its neighbors'. Show that we can get all the lights to be on at the same time.

16. (IMO 2021/6) Let $m \geq 2$ be an integer, A a finite set of integers (not necessarily positive) and $B_1, B_2, ..., B_m$ subsets of A. Suppose that, for every $k = 1, 2, ..., m$, the sum of the elements of B_k is m^k. Prove that A contains at least $\frac{m}{2}$ elements.

17. (USAMO 2008/6) At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form 2^k for some positive integer k).

18. (USAMO 2013/3) Let n be a positive integer. There are $\frac{n(n+1)}{2}$ marks, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n marks. Initially, each mark has the black side up. An operation is to choose a line parallel to the sides of the triangle, and flipping all the marks on that line. A configuration is called admissible if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

6 Even More Problems

What, even more problems? These are problems that can involve linear algebra in some way, but aren't really combinatorics anymore.

19. (IMO Shortlist 2010) Find the smallest number n such that there exist polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \ldots + f_n(x)^2 .$$

20. (HMMT 2017) Let n be an odd positive integer greater than 2, and consider a regular n-gon $\mathcal{G}$ in the plane centered at the origin. Let a subpolygon $\mathcal{G}'$ be a polygon with at least 3 vertices whose vertex set is a subset of that of $\mathcal{G}$. Say $\mathcal{G}'$ is well-centered if its centroid is the origin. Also, say $\mathcal{G}'$ is decomposable if its vertex set can be written as the disjoint union of regular polygons with at least 3 vertices. Show that all well-centered subpolygons are decomposable if and only if n has at most two distinct prime divisors.

21. (Gabriel Dospinescu, 2005) Find all $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ satisfying

$$f(x+y) + f(1) + f(xy) = f(x) + f(y) + f(1+xy)$$

for nonnegative integers x and y.

† 22. (APMO 2017/1 but over $\mathbb{R}$) We call a 5-tuple of real numbers *arrangeable* if its elements can be labeled a, b, c, d, e in some order so that $a - b + c - d + e = 29$. Determine all 2017-tuples of real numbers $x_1, x_2, \ldots, x_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is arrangeable.

23. (APMO 2017/5) Let n be a positive integer. A pair of n-tuples $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ with integer entries is called an exquisite pair if

$$|a_1 b_1 + \cdots + a_n b_n| \leq 1.$$

Determine the maximum number of distinct n-tuples with integer entries such that any two of them form an exquisite pair.

References

A lot of today's content is taken from the following:

- Yufei Zhao's *Algebraic Techniques in Combinatorics* handout, MOP 2007
- Evan Chen's *Linear Algebra Problems* handout, MOP 2017
- Lisa Sauermann's class on *Algebraic Methods in Combinatorics*, MIT 2022